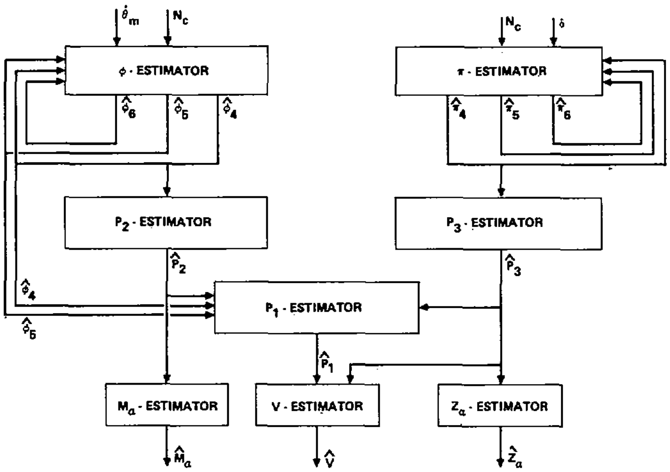

Fig. 2. Four-level estimation technique.

$$\phi_5 = \phi_4 + (C_1C_2 + TP_2)\phi_3 + (C_1C_2C_3 + C_2TP_3 + C_3^3 + TP_1P_2)\phi_2 \tag{36}$$

$$\phi_6 = \phi_5 + (C_1C_2 + TP_2)\phi_4 + (C_1C_2C_3 + C_2TP_3 + TP_1P_2)\phi_3 + (C_1C_2C_3^2 + TP_1^2P_2 + TC_2P_1P_3 + TC_2C_3P_3 + C_3^4)\phi_2. \tag{37}$$

Rather than give all of the details for ϕ_4, ϕ_5, and ϕ_6, we shall first give the details for ϕ_4 and then indicate very briefly how ϕ_5 and ϕ_6 are obtained.

ϕ_4 is obtained by substituting (15), (14), and (5) into the expression

$$\phi_4 = h_{\hat\theta}'\Phi^3\psi. \tag{38}$$

However, $\Phi^2\psi$ is already available from the computation of ϕ_3; it (although we have not derived it here) is

$$\Phi^2\psi = \begin{pmatrix} T\phi_2 \\ \phi_3 \\ C_1\phi_2 + C_3^2C_4 \end{pmatrix}. \tag{39}$$

Thus ϕ_4 is more easily computed as follows:

$$\begin{aligned}\phi_4 = h_{\hat\theta}'\Phi(\Phi^2\psi) &= (0 \quad 1 \quad 0)\begin{pmatrix} P_1 & T & 0 \\ P_2 & 1 & C_2 \\ P_3 & C_1 & C_3 \end{pmatrix}\begin{pmatrix} T\phi_2 \\ \phi_3 \\ C_1\phi_2 + C_3^2C_4 \end{pmatrix} \\ &= (0 \quad 1 \quad 0)\begin{pmatrix} P_1T\phi_2 + T\phi_3 \\ P_2T\phi_2 + \phi_3 + C_1C_2\phi_2 + C_2C_3^2C_4 \\ P_3T\phi_2 + C_1\phi_3 + C_1C_3\phi_2 + C_3^3C_4 \end{pmatrix} \\ &= P_2T\phi_2 + \phi_3 + C_1C_2\phi_2 + C_2C_3^2C_4 \\ &= P_2T\phi_2 + \phi_3 + C_1C_2\phi_2 + C_3^2\phi_2 \end{aligned} \tag{40}$$

where we have used the fact that $C_2C_4 = \phi_2$.

ϕ_5 is computed from the expression

$$\phi_5 = h_{\hat\theta}'\Phi(\Phi^3\psi) \tag{41}$$

where $\Phi^3\psi$ is given by the second matrix in the second line on the right-hand side of (40). Observing that the second row of $\Phi^3\psi$ is ϕ_4, we begin the computation of ϕ_5 by expressing $\Phi^3\psi$ as

$$\Phi^3\psi = \begin{pmatrix} P_1T\phi_2 + T\phi_3 \\ \phi_4 \\ P_3T\phi_2 + C_1\phi_3 + C_1C_3\phi_2 + C_3^3C_4 \end{pmatrix}. \tag{42}$$

The computation of ϕ_5 is completed by carrying through the operations indicated in (41). ϕ_6 is derived in a similar manner.

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On the Structure of Optimal Area Controls in Electric Power Networks

HARRY G. KWATNY AND THOMAS E. BECHERT

***Abstract*—Static optimization techniques have been used by the electric power industry for several years to solve the problem of economic load allocation. Experience has shown that difficulties frequently arise when these solutions are incorporated in the feedback control of dynamic electric power networks. In a recent paper, economic load allocation was formulated as a dynamic optimal control problem in an effort to overcome the disadvantages of controllers currently used. At the heart of that problem is the area control problem that is treated in detail in this paper. An unusual feature of the area control problem is that it contains kinks. The maximal principle is modified for this situation. Necessary conditions for an optimal controller are obtained for the general case of *n* generators. The optimal feedback controller is synthesized for the case of two-generator load sharing.**

I. Introduction

A primary objective in the control of electric power networks is to match generation to load and to distribute the required generation among the available generators in the most economical manner. To accomplish this objective, most modern dispatch controllers determine the steady-state minimum operating cost distribution of load and incorporate the solution to this problem as a trim on the load-frequency controller. Such a procedure often results in unsatisfactory performance, as might be anticipated when the solution to a static problem is imbedded in the feedback control of a dynamic system. Two current trends tend to increase the likelihood of such an occurrence. One is the decrease in system response rate capabilities as a percentage of system capacity. The second is the tendency to include in the dispatch controllers more accurate representation of thermal plant economic characteristics, in particular, recognition of the discontinuous incremental heat rate characteristics of modern multiple valve turbines.

In [1], optimal load allocation is formulated as a dynamic control problem, which takes into account not only the steady-state cost characteristics, but also the dynamic costs involved in changing the level of megawatt generation. The procedure used to design the controller is to partition the overall problem into a single "network control problem" and several identical "area control problems." The feedback controller has been synthesized for the special case of

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two-generator load sharing in one area of a two-area interconnection. In digital simulations, reported in [1], the proposed controller proved far superior to conventional controllers in steering the system rapidly to the new economic operating point following a load change.

In this paper the construction of the optimal area controller is described in detail. Necessary conditions are obtained for the general case in which the area is composed of an arbitrary number of thermal generating stations. These conditions are used to obtain a feedback synthesis of the optimal controller for a two-generator control area. The area control problem has a number of interesting characteristics, and solution of the two-generator problem provides valuable insights into the structure of the general solution.

II. The Area Control Problem

System Model

A control area may be considered to consist of a number of generating stations, each of which is limited to a maximum allowable rate of change of power output [1]. Let

- x_i Power delivered by the ith generating station.
- u_i Rate of change of power output of ith generating station.
- U_i Maximum allowable rate of change of power output of ith generating station.
- n Number of generating stations in the control area.

The dynamic system may be represented by the following set of first-order linear differential equations:

$$\frac{dx_i}{dt} = u_i, \qquad i = 1,2,\ldots,n. \tag{1}$$

The control vector $u(t)$ lies in the restraint set Ω defined by the following:

$$\Omega = \{u\colon |u_i| \leq U_i, \qquad i = 1,2,\cdots,n\}. \tag{2}$$

Performance Evaluation

The function of the area control system is to steer the area generator outputs from an arbitrary initial state to a desired target state which will be specified below. Under normal conditions this can always be accomplished in a finite-time interval which will be designated $[0,T]$. Performance evaluation of candidate controllers will be based on costs incurred within the control area during the transition. These costs include: 1) the duration of the control interval $[0,T]$; 2) the area megawatt error; 3) costs based on the rate of change of power output; and 4) fuel costs. Costs associated with 3) reflect, among other things, reduction of machinery life due to increased mechanical and thermal stresses. The cost functional used in this study is

$$C(u) = \int_0^T \{\alpha_1 + (\Sigma x_j - L)^2 + \alpha_3(\Sigma m_j|u_j|^{q_j}) + \alpha_4(\Sigma h_j(x_j))\}\, dt \tag{3}$$

where L is the total area power demand, $h_j(x_j)$ is the steady-state heat rate characteristic of the jth generating station, and m_j and q_j are constants associated with the jth generating station.

Heat Rate Characteristics

The steady-state fuel consumed per unit time, or heat rate (h), increases with output power generation (x). In most applications, the heat rate characteristics are approximated by smooth convex functions [3], although some applications have explicitly recognized the incremental heat rate discontinuities due to valve points [4],[5]. These discontinuities are considered to be of central importance in the operation of electric power systems and will be included in the present analysis. In this study, the heat rate characteristics will be approximated by piecewise linear curves as shown in Fig. 1.

Target State

Let H denote the overall area heat rate, equal to the sum of the individual station heat rates. Let g denote the area power generation

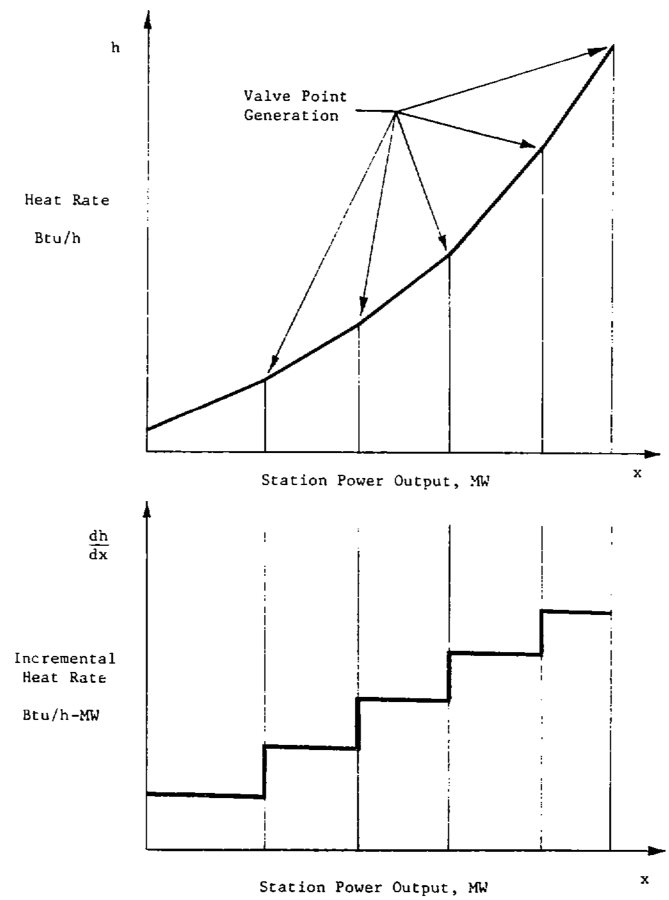


Fig. 1. Approximate heat rate and incremental heat rate characteristics for multiple valve turbine.

deficiency. The target state is defined as the values of $x_1,x_2,\cdots,x_n$ that minimize H, subject to the constraint $x_1 + x_2 + \cdots + x_n = L$.

Statement of the Area Control Problem

The set of admissible controls is the set of measurable control vectors $u(t)$ in Ω that steer the system state (1) from some arbitrary initial state $x(0) = x_0$ to the fixed target state $x(T) = X_T$ corresponding to a fixed load L. The control problem is to find the optimal control $u^*(t)$, i.e., the vector $u(t)$ that steers the system from x_0 to X_T while minimizing the cost functional $C(u)$.

III. Optimal Controller Necessary Conditions

The Maximal Principle

The problem at hand differs from the usual optimal control problem in that the integrand of $C(u)$ contains functions of the state variables, specifically, the unit heat rates $h_i(x_i)$, which have discontinuous first derivatives. It is illuminating to examine the situation in a slightly more general context. Let $f(x)$ be a convex function defined on some open interval D. Then $f(x)$ is supported from below at each point $p \in D$ by a linear support hyperplane

$$S(x,p) = f(p) + \sigma_f(p)(x - p). \tag{4}$$

f is said to be rough at p if it has more than one support hyperplane at p; otherwise, f is said to be smooth at p.

The essential characteristic of the area control problem is that the integrand of $C(u)$ is rough. Luenberger [2] has applied the term "kinks" to describe such a situation. In the presence of kinks the usual theorems that provide necessary conditions for optimal control fail to apply. The approach taken here will be to appropriately modify the maximal principle in order to obtain a suitable set of necessary conditions. As will be seen, the required modifications are straightforward. Nevertheless, they have distinctive consequences on the final solution.

Consider the linear time-invariant control process in R^n

$$\dot{x} = Ax + Bu(t)$$

with cost functional

$$C(u) = \int_0^T \{f^0(x) + h^0(u)\}\, dt$$

where $f^0(x)$ and $h^0(u)$ are positive convex functions, and with compact, convex control restraint $u(t) \subset \Omega \subset R^m$. The state vector $x(t)$ is extended to an $(n+1)$-vector

$$\hat{x}(t) = (x_0(t), x_1(t), \cdots, x_n(t))'$$

by defining an additional state variable

$$\dot{x}_0 = f^0(x) + h^0(u), \qquad x_0(0) = 0.$$

An $(n+1)$-dimensional augmented adjoint vector $\hat{\eta}(t)$ is defined as a continuous solution of the system of equations

$$\eta_0(t) = \text{constant}$$

$$\dot{\eta} = -\eta A - \eta_0 \sigma_{f^0}(x), \qquad \eta \in R^{n*}.$$

The Hamiltonian function is defined as

$$H(\hat{\eta},\hat{x},u) = \eta_0(f^0(x) + h^0(u)) + \eta'(Ax + Bu).$$

The maximum value of the Hamiltonian over all values of u in the restraint set Ω is denoted by M, i.e.,

$$M(\hat{\eta},\hat{x}) = \max_{u \in \Omega} H(\hat{\eta},\hat{x},u).$$

All measurable controls $u(t) \subset \Omega$ on finite intervals $[0,T]$ that steer the system from an initial state $x(0) = x_0$ to a final state $x(T) = X_T$ are admissible. The following theorem provides the required necessary conditions for optimality.

Theorem: If $u^*(t)$ is an admissible control with response $x^*(t)$ that minimizes $C(u)$, then it is necessary that

1) there exist a nontrivial augmented adjoint response $\hat{\eta}^*(t)$ such that
2) $H(\hat{\eta}^*,\hat{x}^*,u^*) = M(\hat{\eta}^*,\hat{x}^*)$ almost everywhere on $[0,T]$,
3) $M(\hat{\eta}^*,\hat{x}^*) = 0$ everywhere on $[0,T]$, and
4) $\eta_0 < 0$.

Note that, if $f^0(x)$ is smooth, then $\sigma_{f^0}(x) = \partial f^0/\partial x$ and the above theorem is the usual maximal principle. Proof of the theorem in its present form follows easily with minor modification of standard techniques (in particular, see Chapter 3 of Markus and Lee [6]).

In the area control problem, the following identifications can be made:

$$f^0(x) = \alpha_1 + \{\Sigma x_j - L\}^2 + \alpha_4\{\Sigma h_j(x_j)\} \tag{5}$$

$$h^0(u) = \alpha_3\{\Sigma m_j u_j^{q_j}\} \tag{6}$$

$$H(\hat{\eta},\hat{x},u) = \eta_0\{f^0(x) + h^0(u)\} + \Sigma \eta_j u_j. \tag{7}$$

Also, the ith component of σ_{f^0} is given by

$$(\sigma_{f^0})_i = 2\{\Sigma x_j - L\} - 2\sigma_i, \qquad i = 1,\cdots,n$$

where

$$\sigma_i = \frac{\alpha_4}{2}\frac{dh_i}{dx_i}$$

and where dh_i/dx_i is to be interpreted to assume any value between the left- and right-hand derivatives at a point of discontinuity. Consequently, the adjoint equations are

$$\dot{\eta}_i = -2\eta_0\{\Sigma x_j - L\} - 2\eta_0\sigma_i, \qquad i = 1,\cdots,n. \tag{8}$$

The conditions of the maximal principle are still satisfied if the adjoint vector $\hat{\eta}^*$ is replaced by $\hat{\eta}^*/\eta_0$. Without loss of generality, it is henceforth assumed that $\eta_0 = -1$.

Maximization of the Hamiltonian

Denoting $\beta_i = \alpha_0 m_i$, maximizing the Hamiltonian as given by (5)–(7) with respect to u is equivalent to maximizing each of the functions

$$G_i(u_i) = \eta_i u_i - \beta_i |u_i|^{q_i}, \qquad i = 1,\cdots,n$$

with respect to u_i. If $q_i > 1$, then it is not difficult to show that

$$u_i^* = \begin{cases} U_i & \Delta > U_i \\ \Delta & |\Delta| \le U_i \\ -U_i & \Delta < -U_i \end{cases} \tag{9}$$

where

$$\Delta = \left[\frac{|\eta_i|}{\beta_i q_i}\right]^{\frac{1}{q_i - 1}} \operatorname{sgn} \eta_i.$$

If $q_i = 1$, then

$$u_i^* = \begin{cases} U_i & \eta_i > \beta_i \\ 0 & |\eta_i| < \beta_i \\ -U_i & \eta_i < -\beta_i. \end{cases} \tag{10}$$

The special points $\eta_i = \pm\beta_i$ are of interest in the event that the adjoint response $\eta_i(t)$ might dwell at one of these points for a finite-time interval. If on a finite-time interval $\eta_i(t) \equiv +\beta_i$, then G_i is maximized [with $G_i(u_i) = 0$] by all u_i in the range $0 \le u_i \le U_i$. Similarly, if $\eta_i(t) \equiv -\beta_i$, G_i is maximized by all u_i in the range $-U_i \le u_i \le 0$. These points are "singular points"; here, maximizing the Hamiltonian does not uniquely define the optimal control [7], [8]. Additional necessary conditions can be obtained from the state and adjoint equations.

Suppose, for the first k of the n generating stations, the adjoint response is $\eta_i \equiv \pm\beta_i$ on some finite interval (a,b). Then, the optimal control u_i^* is well defined for $i = k+1,\cdots,n$, but is singular for $i = 1,\cdots,k$. On (a,b), according to the adjoint canonical equations, for $i = 1,\cdots,k$,

$$\dot{\eta}_i \equiv 0 = 2[\Sigma x_j - L] + \alpha_4 \frac{dh_i(x_i)}{dx_i}, \qquad k < n.$$

Thus, these k stations all operate with the same value of incremental heat rate, which is proportional to the area megawatt deficiency. Also, on (a,b), for $i = 1,\cdots,k$, $\ddot{\eta}_i \equiv 0$, which leads to

$$\Sigma u_j + \frac{\alpha_4}{2}\frac{d^2 h_i(x_i)}{dx_i^2} u_i = 0. \tag{11}$$

Since a staircase function has been assumed for the incremental heat rate characteristic, the second derivative vanishes (except at a valve point, where it is undefined). Hence, the following relation holds:

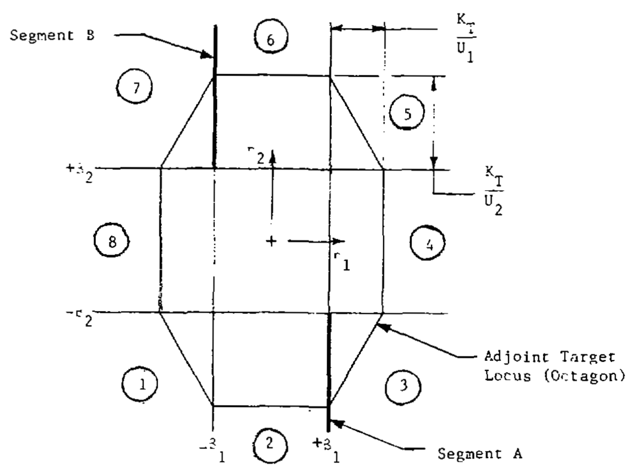
$$\Sigma_1^k u_j = -\Sigma_{k+1}^n u_j. \tag{12}$$

Thus, the summation of the singular controllers must equal the negative summation of the nonsingular controllers. It may happen that the boundedness of the controllers prevents fulfillment of this condition. This means that the first j adjoint equations cannot be satisfied on the finite-time interval (a,b) by the solutions $\eta_i = \pm\beta_i$, $i = 1,\cdots,k$, and hence such a singular condition cannot exist.

For the case $k = 1$, that is, when only one of the n adjoint variables dwells on $\eta = \pm\beta$, on the finite-time interval (a,b), then the optimal singular control is uniquely specified on (a,b) by (12).

Adjoint Vector Boundary Conditions

The conditions of the maximal principle can be applied to obtain necessary conditions on the initial and final values of the adjoint variables. Consider the condition $H(\hat{\eta},\hat{x},u) = 0$, almost everywhere



Zone	u_1^*	u_2^*
1	$-U_1$	$-U_2$
2	0	$-U_2$
3	$+U_1$	$-U_2$
4	$+U_1$	0
5	$+U_1$	$+U_2$
6	0	$+U_2$
7	$-U_1$	$+U_2$
8	$-U_1$	0
Segment A	$+U_2$	$-U_2$
Segment B	$-U_2$	$+U_2$

Fig. 2. Definition of control zones in adjoint space.

along optimal trajectories. In particular, for $t = 0^+$, and $t = T^-$, the optimal state response $x(t)$ is equal to x_0 and x_T, respectively, and the corresponding boundary values of the adjoint vector are denoted η_0 and η_T. These boundary values of the state vector and adjoint vector are related by the equation

$$\Sigma[\eta_i u_i - \beta_i |u_i|^{q_i}] = \alpha_1 + \{\Sigma x_j - L\}^2 = \alpha_4\{\Sigma h_j(x_j)\}. \quad (13)$$

For the initial and terminal times, the right side of (13) depends only on the known vectors x_0 and x_T. The left side of (13) depends only on the optimal values of η_i and u_i. But in the previous sections, relations were derived that express the optimal controllers u_i^* in terms of the optimal adjoint variables η_i^*. Therefore, the left side of (13) depends only on the optimal adjoint vector η_i^*. Solutions of (13) are hypersurfaces in the n-dimensional adjoint space representing the locus of allowable initial and target vectors η_0 and η_T, respectively.

For one special case of interest, when $q = 1$, $n = 2$, the resulting hypersurface is the polygon shown in Fig. 2.

IV. Solution for Special Case

The optimal feedback controller will be synthesized for the case in which the control area contains just two generating stations ($n = 2$), and the cost functional penalizes the control input to the first power ($q = 1$). For convenience, the station with the larger maximum control bound is designated Number 1 ($U_1 > U_2$). The Hamiltonian maximization conditions provide the results summarized in Fig. 2. Singular solutions exist on Segments A and B.

The feedback controller is synthesized by integrating the state and adjoint equations backward in time. By starting at $t = T$, with the state vector (x_1,x_2) at the target point (x_{1T},x_{2T}) and with the adjoint vector at some arbitrary point on the target octagon, an optimal

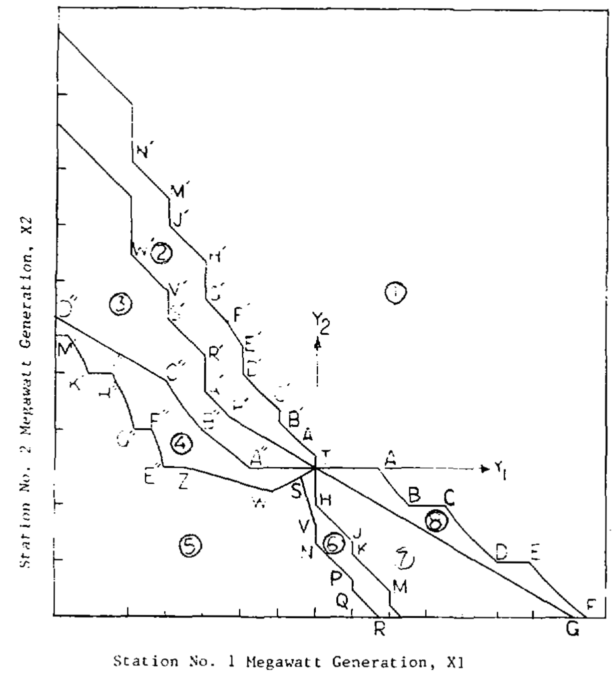


Fig. 3. Overall state portrait—switching curves.

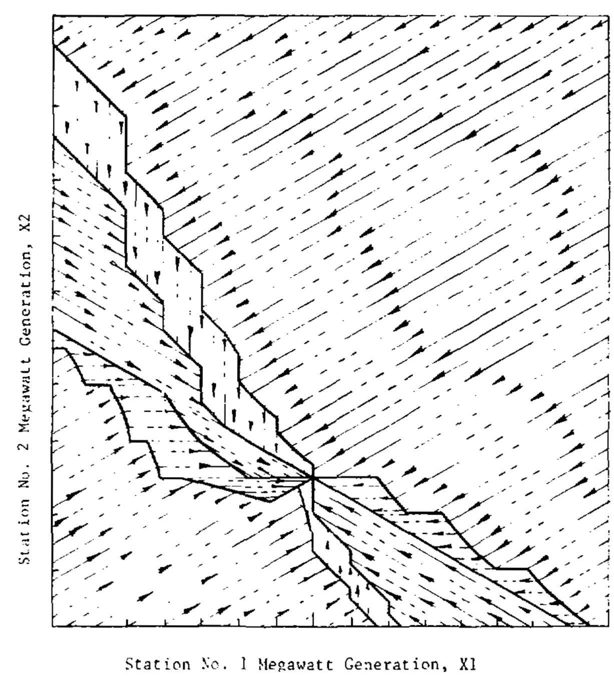


Fig. 4. Overall state portrait—optimal trajectories.

trajectory may be traced out in the state and adjoint spaces for all subintervals with $t < T$. This process may be repeated for all points on the target octagon, thereby tracing out all possible optimal trajectories.

In particular, the points at which optimal trajectories cross the controller switching lines ($\eta_1 = \pm\beta_1$, $\eta_2 = \pm\beta_2$) may be mapped into the state space. When all such switching lines have been mapped into the state space, the state space will have been divided into regions of constant optimal control action (u_1^*,u_2^*). This will complete the synthesis of the optimal feedback controller.

The switching lines obtained in this way are shown in Fig. 3. The control values in each zone are tabulated in Fig. 2. Optimal trajectories in the state space are shown in Fig. 4. Details of the computation of the switching lines can be found in [10].

V. Valve Point Singularities

An important characteristic of the optimal controller is observed by examining the corresponding optimal adjoint and state trajectories illustrated in Figs. 5 and 6, respectively.

It is noted that Fig. 5 shows two trajectories terminating at T_8, namely, $(0_8, N, M, T_8)$ and $(0_8', P, M, T_7)$. The corresponding state-space trajectories are shown in Fig. 6. This situation arises when a trajectory dwells on a Number 2 valve point over a finite-time interval, as illustrated by the adjoint trajectories through point M. The question arises: What is the correct value of σ_2 while the trajector passes through Zone 8? In order to travel from N to M, the value $\sigma_2 = \sigma_{24}$ must be used. In order to travel from P to M, the value $\sigma_2 = \sigma_{23}$ must be used. But for trajectories that enter Zone 8 at points N' between P and N, the optimal value of σ_2 is not uniquely determined. As shown by Fig. 7, neither σ_{23} nor σ_{24} will lead from N' to M. One approach, illustrated in Fig. 7, would be to set $\sigma_2 = \sigma_{24}$ on (N', P') and then to switch to $\sigma_2 = \sigma_{23}$ on (P', M). This approach would lead to an optimal trajectory to the target T_8. But an equally effective approach would be to set $\sigma_2 = \sigma_{23}$ at point N', and then to switch back to σ_{24} when the trajectory intersects the curve NM. Or, intermediate values of σ_2, between σ_{23} and σ_{24}, could be chosen such that the trajectory arrives at the $+\beta_2$ switching line at time $t = M$. In all these approaches, the state-space trajectories are identical; they travel along the valve point from N' to point M, regardless of the approach chosen for selecting the value for σ_2.

It is significant that the choice of σ_2 does not affect the value of the cost functional. The overall time interval T, the first term of the cost functional, is determined only by U_1 in this case, and is unaffected by σ_2. The second and third terms of the cost functional contain the time functions x_1, x_2, u_1, and u_2, which are unaffected by σ_2. And the final cost term depends on the *total* fuel cost curve, not the *incremental* cost curve. For values of σ_2 between σ_{23} and σ_{24}, this term of the cost functional is unaffected.

Hence, any value (or sequence of values) for σ_2 that leads the trajectory from N' to M will be acceptable, since neither the state-space trajectories, nor the switching loci, nor the value of the cost functional will be affected by the choice. It is interesting to note that the approach taken by Luenberger [2] specifies $\sigma_2(t)$ through an associated optimal control problem. The state-space switching hyperbolas are therefore joined together by line segments along the valve points, such as the segment $\overline{NP}$.

VI. Conclusions

This paper has reported a new formulation of the area control problem. In essence, the proposed controller combines the functions of "economic dispatch" and "regulation" in a meaningful way. The control problem is formulated for the control area, placing an upper bound on the allowable rate of change of power output for each generator and incorporating a performance measure that includes consideration of: time to target; area fuel costs; area megawatt error; and rate of change of generation. This dynamic optimal control problem is solved to find the optimal rate at which each generator should be driven toward its megawatt target, such that the cost functional is minimized.

Throttling losses, present in each valve region of multiple valve turbines, are considered throughout this research. A staircase function is used to represent the incremental cost curve, reflecting this characteristic. It should be noted that the resulting target state has the property that all generating stations operate at a valve point except one, whose output trims the total generation to match demand. In the dynamic situation small changes in demand affect only this generator; the others remain at constant output. It is interesting to note that it is primarily the inclusion of fuel costs that holds the remaining generators at constant output, although the rate of change of generation penalty produces a similar, but lesser, effect. Larger deviations, of course, bring in the second generator to assist in reducing the megawatt error.

The artificial distinction between regulating and economic generat-

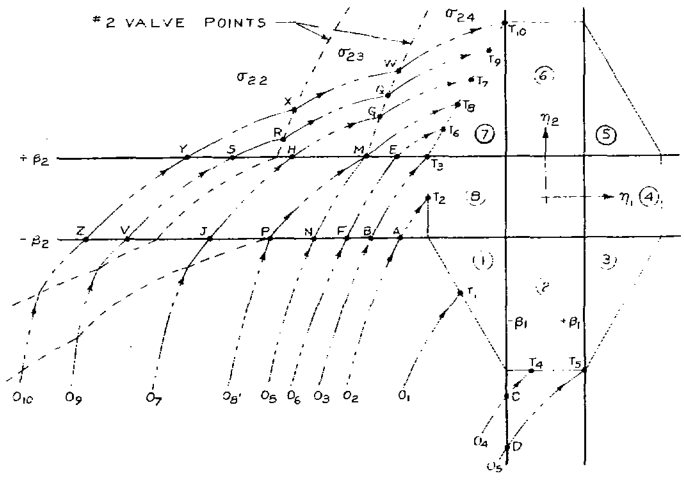


Fig. 5. Optimal adjoint trajectory targets in Zones 7, 8, 1, and 2.

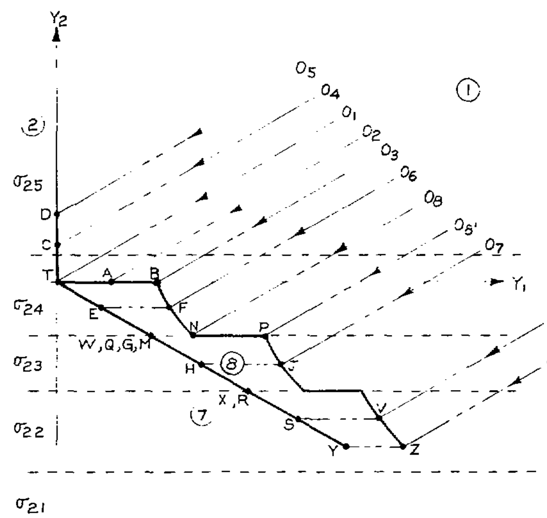


Fig. 6. Optimal state trajectory targets in Zones 7, 8, 1, and 2.

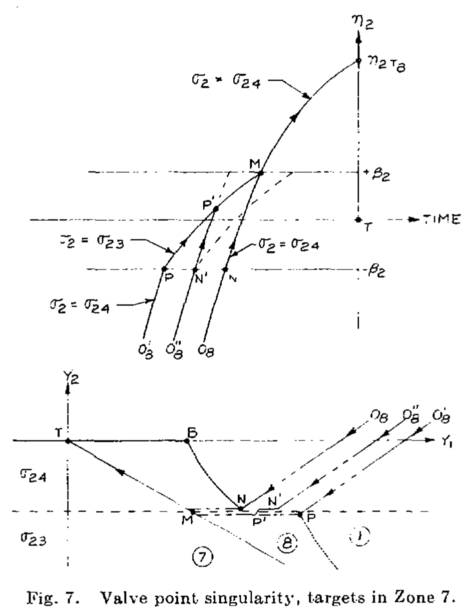


Fig. 7. Valve point singularity, targets in Zone 7.

ing units is eliminated as all generators on automatic control are available for load tracking whenever it is necessary to use them. On the other hand, only the most advantageous generators are maneuvered, depending upon the system state and load demand, as well as the individual generator dynamic and economic characteristics. A related affect, due to the incremental heat rate discontinuities (cost functional kinks), is that certain trajectories tend to dwell at various valve points encountered en route to the target.

Necessary conditions for the optimal controller were derived for an arbitrary number of participating generators. A feedback controller was synthesized for the special case of two-generator load sharing and was characterized by specification of the switching lines in the state space. This procedure would not be suitable for systems involving a greater number of generating stations because of the difficulty in storing the complex switching surfaces, even if they could be obtained. A numerical procedure for computer solution of the state and adjoint differential equations is a reasonable alternative. Standard procedures do not apply, however, because of the singular nature of the solutions. Although the controller singularities are easily remedied, the valve point singularities are not.

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A Maximal-Order Theorem for Optimal Rational Models

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***Abstract*—The problem of finding an optimal approximating model (in L_2-sense) to a fixed filter out of the class of rational filters of "order" $\leq k$ is considered. After the existence question is settled, it is shown that such an optimal model must in a certain sense be of maximal-order k.**

I. Introduction

In the numerical-analysis approximation theory literature, the past decade has seen a good deal of interest devoted to the rational approximation problem—more explicitly, to the problem of optimally approximating a fixed (most often continuous) function on a finite-interval or finite-measure space by a function selected from some subclass of the rational functions (or what have been called "generalized rational" functions). The names associated with this work are Cheney, Loeb, Goldstein, and Walsh, among others (see [1]–[4]).

It is clear to the reader oriented more toward system theory that an analog to the rational approximation problem exists in linear system theory. Namely, given a causal[1] convolution filter G (not necessarily with rational transfer function), one can consider the problem of "optimally modeling" G by a causal[1] filter r with rational transfer function of "order" $\leq k$, i.e., by a finite-order system of order $\leq k$. This is the problem on which we shall focus.

Following Cheney and Goldstein [1], we make use of the L_2-norm (rather than the Chebyshev norm as in [2] and [3]). However, the nature of our problem forces us to consider integrals over the noncompact domain $(-\infty, \infty)$. Also, the functions involved in our study are more stringently restricted than in [1] since they represent causal[1] filters. As such, they have Laplace transforms analytic in the right-half plane.

Having said the above, we now add that analogs of some of the results of [1] do, in fact, carry over to our context. The methods of [1] do not carry over due to the noncompactness of our domain. However, we do have additional analyticity properties at our disposal, and these are brought into play to yield our results.

In Section III an affirmative answer is given to the existence question for optimal rational models of restricted order. The difficulty is that the rational functions do not form a linear space, so the "usual" techniques do not work. Our method consists of transforming the problem into an equivalent one on the unit disk. Section II deals with the preliminaries required to effect this transformation.

In Section IV, Theorem 2, we prove the main result of this paper: If G does not have a *rational* transfer function, then an optimal rational model for G selected from the class of *rationals* of order $\leq k$ must actually have full-order k. (Take order to mean order of denominator when in lowest terms.)

II. Mathematical Preliminaries

Definition 1: Let L_2^+ denote the Hilbert space of real-valued measurable functions on $(-\infty, \infty)$ which are square integrable with respect to Lebesgue measure, and which have support in $[0, \infty)$. We shall informally refer to elements of L_2^+ as *causal L_2-filters.*

Definition 2: For $g(\cdot)$ a causal L_2-filter $(g(\cdot) \in L_2^+)$ we define the Fourier-Laplace transform $G(s)$ of $g(\cdot)$ by

$$G(s) = \int_0^\infty \exp(-st)\, g(t)\, dt, \qquad \text{for } s \text{ with } \operatorname{Re}\{s\} \geq 0;$$

$G(i\omega)$ is thus the L_2-Fourier transform of $g(\cdot)$.

Definition 3: For each integer $k > 0$ let R_k denote the set of rational functions of the complex variable s writable in lowest terms as $q(s)/p(s)$ where

a) $o(q(s)) < o(p(s)) \leq k$,

$$o(p(s)) = \text{degree of the polynomial } p(s), \text{ etc.};$$

b) $p(s)$ has no zeros in $\operatorname{Re}\{s\} \geq 0$.

Remark 1: Each element $r(s)$ of R_k is the Fourier-Laplace transform of some causal L_2-filter $h(t)$.

The next section will treat the existence question for our approximation problem by transforming it into an equivalent problem defined on the unit circle. In the remainder of this preliminary section we study the transformation required.

In this direction, we now consider the linear fractional transformation $z = v(s) = (1 - s)/1 + s$, which maps the right-half plane $\operatorname{Re}\{s\} > 0$ holomorphically $1 - 1$ onto the unit disk $|z| < 1$ while also taking the imaginary axis to $|z| = 1$, i.e.,

$$\exp(i\theta) = \frac{1 - i\omega}{1 + i\omega}, \qquad \text{for } -\infty < \omega < \infty$$

and $-\pi < \theta < \pi$ (while $\omega = \pm\infty$ corresponds to $\theta = \pm\pi$).

Definition 4: For each integer $k > 0$ we shall denote by R_1^k the set of rational functions of the complex variable z defined on $|z| \leq 1$, which can be written in lowest terms as $q(z)/p(z)$ where

a) $o(p(z)) \leq k$ and $o(q(z)) \leq k$;
b) $p(z)$ has no zeros in $|z| \leq 1$;
c) $q(-1) = 0$.

The relationship between R_k and R_1^k is easily stated. We state

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[1] For definition and properties of causality, see [6]–[8].